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Physical geodesy of Neumann's boundary—value problem for GPS—based gravimetries

By

Yukio HAGIWARA*

National Research Institute for Earth Science and Disaster Prevention, Japan

Abstract

Recent development of the GPS satellite technique has made it possible to determine the three—dimensional geodetic position of points on the Earth's surface. In the conventional way of geodesy the geoid was a basis for determining the geodetic position. At that time we dealt with the geoid—based theory of physical geodesy using the Stokes and the Vening—Meinesz integral formulas with gravity anomaly. In the satellite age another quantity, "gravity disturbance", which is defined directly on the Earth's surface, is used instead of gravity anomaly. The Neumann boundary—value problem and its inverse problem are formulated here using the Neumann and the modified Vening—Meinesz integral formulas with gravity disturbance, estimating the truncation error of numerical integrations, and taking the Molodenskii terrain correction terms into consideration. Furthermore, the Cartesian coordinate approximation for practical calculations of GPS—based physical geodesy is introduced.

Key words: gravity disturbance,—GPS (Global Positioning System), Neumann's boundary —value problem, physical geodesy.

1. Introduction

One of the major problems that faced physical geodesists was the determination of the geoid from a set of geodetic and gravity data, until satellite techniques became applicable for geodetic measurements. Until that time geodetic measurements were reduced to the geoid by solving the geodetic boundary—value problem through the Stokes and Vening—Meinesz integrals. The geoid was then a basis for determining geodetic positions on the Earth's surface. However, the geoid is an imaginary surface in land areas as defined by an equipotential extension landward from the quasi—stationary sea surface. Despite the fact that the theoretical aspect of physical geodesy was made clear by the Molodenskii approach (Molodenskii et al., 1962), some ambiguities remained in the practical determination of the geoid.

Recent development of satellite techniques, such as GPS (Global Positioning System), has made it possible to determine accurately the three—dimensional geodetic position of points on the Earth's surface. The advantage of this satellite method is that

* Director—General

such a determined position is directly referred to the geocentric coordinates without any consideration of the geoid. The geoid—based concept of gravity anomaly is being replaced due to the new satellite system by another quantity, "gravity disturbance" (Heiskanen and Moritz, 1967), which is defined directly on the Earth's surface and should be used instead of gravity anomaly. Thus, the conventional way of solving the Stokes and Vening—Meinesz integrals with gravity anomaly should also be replaced by the solution of the Neumann boundary—value problem for gravity disturbance.

The purpose of this paper is to obtain a new formulation of gravity disturbance by rewriting conventionally—used mathematical formulas relating to the gravity anomaly. The Neumann boundary—value problem and its inverse problem with the Neumann and the modified Vening—Meinesz integrals, estimating the truncation errors of numerical integrations, and taking the Molodenskii terrain correction terms into consideration will be discussed herein. In addition, the use of spectral analysis techniques available for data gridded in the Caresian coordinates, with their applications to the evaluations of the Neumann and the modified Vening—Meinesz integral formulas and the Molodenskii terrain correction terms will be considered. It is assumed that the method and formulas originally introduced in this paper can be used for practical calculations of GPS—based physical geodesy.

2. New Height System and Gravity Disturbance

First of all, the geodetic height system according to the modern theory of physical geodesy will be considered. First, suppose a point P on the Earth's surface and another point R on the ellipsoid, as shown in Fig. 1. The straight line PR, which is normal to the ellipsoid, is called the true height of P above the ellipsoid. The direct measurement of the true height can not be made by the leveling survey method, but in recent years its highly accurate determination can be realized by a satellite distance—measuring technique such as GPS.

A new surface called the telluroid is drawn between the Earth's surface and the ellipsoid in Fig. 1. The true height PR ($=h$) is then divided by the telluroid into two parts: normal height QR ($=H$) and height anomaly PQ ($=\zeta$). The normal height is derived from geopotential differences and is measured in practice by the leveling combined with gravity measurements. The conventional leveling height is not exactly equal to the normal height. In the first approximation theory, the leveling height with a correction considering geopotential differences can be practically used for the normal height.

The height anomaly is defined as the height difference between the true height and the normal height. In the conventional way, the height anomaly is obtained by numerical computations of the Stokes integral with gravity anomaly over the whole surface of the Earth. The modern determination of the height anomaly can be made by comparing the normal height with the GPS—derived true height (Engelis et al., 1984 ; Denker and Wenzel, 1987). From this one can see that the ambiguous concept of "geoid" in old—fashioned geodesy is being replaced by this new height system.

Consider now the gravity g measured at P on the Earth's surface. Meanwhile, the normal gravity γ at the same point is computed by using Somigliana's normal gravity

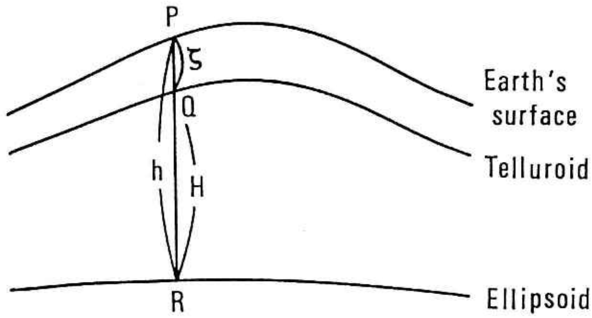

Fig. 1 Height system definition with normal height H, height anomaly ζ and true height $h(=H+\zeta)$.

formula and the vertical gradient of normal gravity. The gravity disturbance is then defined as their differences:

$$\delta g = g(P) - \gamma(P). \tag{1}$$

g at P is measured by a gravimeter, while γ is computed if the geodetic position of P is determined by GPS. Therefore, δg is an obtainable quantity.

Up to the time when the GPS method became available, the true height could not actually be obtained, so that the gravity disturbance was an imaginary quantity. Instead of the gravity disturbance, the gravity anomaly is defined as

$$\Delta g = g(P) - \gamma(Q). \tag{2}$$

As the normal height of Q is measured by leveling and gravity measurements, we can compute γ at Q and then obtain Δg.

The disadvantage of this definition, however, lies in the fact that the geodetic positions of P and Q are different from each other. The height anomaly PQ remains unknown until one performs the Stokes integration of obtained gravity anomalies. Despite such a disadvantage, old-fashioned textbooks of physical geodesy intensified the importance of the gravity anomaly because it was an obtainable quantity. However, the recent development of the GPS method has made a radical change in geodetic importance, and the gravity disturbance has thus replaced the importance of gravity anomaly.

As a matter of fact, many are very familiar with the term "gravity anomaly" in fields of both geodesy and geophysics. This term will survive even after gravity disturbance takes its place. Yet my personal opinion is that we should abolish the conventional definition of gravity anomaly in the form of Eq. (2) and adopt a newly defined "gravity anomaly" as in Eq. (1) instead of using the term "gravity disturbance". I think that this will be a way to avoid confusion at the time of the revolution of geodesy.

3. Neumann's Boundary-Value Problem

Suppose a potential function T (r, ϕ, λ) outside a certain closed surface S, where r, ϕ and λ are distance, geodetic latitude and longitude in the spherical coordinates. When T satisfies Laplace's equation

$$\nabla^2 T = 0 \tag{3}$$

outside S, it is called harmonic in the exterior of S. There always exists boundary values of T on S. It is possible to compute values of T at every point outside S from known boundary values on S. This mathematical procedure is called "Dirichlet's problem" or the "first boundary—value problem of potential theory".

The second boundary—value problem, alias "Neumann's problem", is defined as follows: when the normal derivative of T is given on S, one can compute T at every point in the exterior of S. The gravity disturbance δg is a normal derivative of T. If the shape of the Earth is approximated by a sphere with the radius R, the boundary —value condition is given by

$$\delta g = -\left[\frac{\partial T}{\partial r}\right]_{r=R} \tag{4}$$

at a certain point P on S. Eq. (4) indicates that the gravity disturbance obeys Neumann's problem.

In the third boundary—value problem a linear combination of T and its normal derivative is given on S. The definition of gravity anomaly includes implicitly height anomaly ζ (PQ in Fig. 1). In this case, a correction term for the discrepancy of ζ should be added to the boundary—value condition. For a spherical approximation of the Earth's shape, the boundary—value condition of the gravity anomaly can be expressed by

$$\Delta g = -\left[\frac{\partial T}{\partial r} + \frac{2T}{r}\right]_{r=R} \tag{5}$$

instead of Eq. (4). It is seen that Eq. (5) has a form of the third boundary—value problem, consisting of a linear combination of T and its normal derivative.

Consider now a solution of Neumann's problem expressed by a spherical surface harmonic function. T is expressed by a spherical surface harmonic series as

$$T(r, \phi, \lambda) = \frac{GM}{R}\sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^{n+1} Y_n(\phi, \lambda), \tag{6}$$

with the spherical surface harmonic function as

$$Y_n(\phi, \lambda) = \sum_{m=0}^{n}(C_n^m \cos m\lambda + S_n^m \sin m\lambda) P_n^m(\sin\phi), \tag{7}$$

where G is Newton's gravitational constant, M the Earth's mass, P_n^m the associated Legendre function, and C_n^m and S_n^m spherical harmonic coefficients. It can be easily proved that Eq. (6) satisfies Laplace's Eq. (3).

Notice that Eq. (6) excludes terms of degree $n=0$ and $n=1$. Assume the mean value of T to be zero, so that the zero—degree term is not included in Eq. (6). The first degree terms correspond to the coordinates of the Earth's gravity center. The origin of the spherical coordinates is set at the Earth's gravity center. Therefore, the first—degree terms vanish automatically from Eq. (6).

Differentiating Eq. (6) with respect to r and substituting it into Eq. (4), we have

$$\delta g(\phi, \lambda)=\sum_{n=2}^{\infty}\delta g_n(\phi, \lambda), \tag{8}$$

where

$$\delta g_n(\phi, \lambda)=\frac{GM}{R^2}(n+1)Y_n(\phi, \lambda). \tag{9}$$

Eliminating $Y_n(\phi, \lambda)$ from both Eq. (6) and Eq. (9), the potential disturbance T is in the form of the summation of the spherical surface harmonic function of gravity disturbance δg_n, that is

$$T(r, \phi, \lambda)=R\sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^{n+1}\frac{\delta g_n(\phi, \lambda)}{n+1}. \tag{10}$$

Taking two points P and P' in the coordinates (ϕ, λ) and (ϕ', λ') on the spherical surface S, ψ is denoted as the spherical angle between P and P′, which is given by the cosine formula,

$$\cos\psi=\sin\phi\ \sin\phi'+\cos\phi\ \cos\phi'\cos(\lambda'-\lambda), \tag{11}$$

(see Fig. 3). Applying the orthogonality condition of the associated Legendre function to both sides of Eq. (8), we obtain an integral form :

$$\delta g_n(\phi, \lambda)=\frac{2n+1}{4\pi}\int_0^{2\pi}d\lambda'\int_0^{\pi}\delta g(\phi', \lambda')P_n(\cos\psi)\cos\phi' d\phi'. \tag{12}$$

Substitution of Eq. (12) into Eq. (10) gives the Neumann integral

$$T(r, \phi, \lambda)=\frac{R}{4\pi}\int_0^{2\pi}d\lambda'\int_0^{\pi}N(r, \psi)\delta g(\phi', \lambda')\cos\phi' d\phi', \tag{13}$$

where

$$N(r, \psi)=\sum_{n=2}^{\infty}\frac{2n+1}{n+1}\left(\frac{R}{r}\right)^{n+1}P_n(\cos\psi). \tag{14}$$

We call $N(r, \psi)$ Neumann's function or the modified Stokes function. Eq. (13) shows that, if δg is obtained by gravity measurements on S, values of T at every point outside S can be computed by integrating δg weighted with $N(r, \psi)$. This is the solution of Neumann's problem for gravity disturbance. Practically Eq. (13) can be evaluated by a summation over finite compartments dividing the integration range.

It is very convenient for the numerical integration of Eq. (13) to rewrite $N(r, \psi)$ as a closed formula. Using a summation formula of Legendre's polynomial with $x=\cos\psi$

$$\sum_{n=0}^{\infty}\frac{2n+1}{n+1}t^{n+1}P_n(x)=\frac{2t}{\sqrt{1-2xt+t^2}}-\log\frac{t-x+\sqrt{1-2xt+t^2}}{1-x}, \tag{15}$$

the closed formula of the Neumann function is then obtained in the form :

$$N(r, \psi)=-\frac{R}{r}-\frac{3}{2}\left(\frac{R}{r}\right)^2\cos\psi+\frac{2R}{\ell}-\log\frac{R-r\cos\psi+\ell}{r(1-\cos\psi)}, \tag{16}$$

where

$$\ell=\sqrt{r^2+R^2-2Rr\cos\psi}. \tag{17}$$

The first and second terms of the righthand side of Eq. (16) correspond to the zero—and first—degree terms of the lefthand side of Eq. (15).

If the boundary values of T on the surface S are known, the height anomaly can be estimated. Using the Bruns formula, which relates the height anomaly to the potential disturbance, from Eq. (13) we have

$$\zeta(\phi,\lambda)=\frac{T(R,\phi,\lambda)}{\gamma}$$

$$=\frac{R}{4\pi\gamma}\int_0^{2\pi}d\lambda'\int_0^{\pi}N(R,\psi)\delta g(\phi',\lambda')\cos\phi' d\phi'. \tag{18}$$

As previously mentioned, the height anomaly is obtained directly by subtracting the leveling height from the GPS—derived true height. Eq. (18) shows that it is also possible to determine the height anomaly from the gravity disturbance. The comparison of these two kinds of height anomalies enables one to check the accuracies of the geodetic measurements. Eq. (18) is also called Neumann's integral. Taking $r=R$ in Eq. (16), we get

$$N(R,\psi)=-1-\frac{3}{2}\cos\psi+\operatorname{cosec}\frac{\psi}{2}-\log\left(1+\operatorname{cosec}\frac{\psi}{2}\right). \tag{19}$$

Almost similarly the solution of the third boundary—value problem for gravity anomaly can be obtained. However, as this solution is given in almost all textbooks of physical geodesy, it is omitted here. In this case, $N(r,\psi)$ is replaced by the Stokes function

$$S(r,\psi)=\sum_{n=2}^{\infty}\frac{2n+1}{n-1}\left(\frac{R}{r}\right)^{n+1}P_n(\cos\psi). \tag{20}$$

It is interesting to notice that the denominator $n+1$ in Eq. (14) is replaced by $n-1$ in Eq. (20). Since both the functions $N(R,\psi)$ and $S(R,\psi)$ diverge to infinity when $\psi=0$, they are multiplied by $\sin\psi$ and shown in Fig. 2. $N(R,\psi)\sin\psi$ and $S(R,\psi)\sin\psi$ converge to 2 when $\psi=0$.

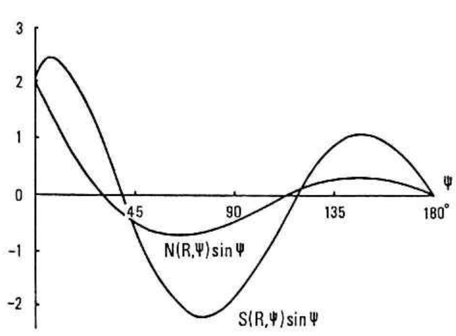

Fig. 2 The Neumann function $N(R,\psi)\sin\psi$ and the Stokes function $S(R,\psi)\sin\psi$.

4. Modified Vening—Meinesz Formula

The vertical PR is normal to the ellipsoid in Fig. 1. The direction of gravity, which is known as the plumbline, at an observation point P differs slightly from the direction of the vertical. The included angle between the vertical and the plumbline is called the deflection of the vertical or the deflection of the plumbline. This small angle is related to the horizontal gradient of the height anomaly. A north—south component and an east—west component of the deflection of the vertical, denoted by ξ and η, are given—by differentiating the height anomaly ζ with respect to ϕ and λ, that is

$$\left.\begin{aligned} \xi &= -\frac{1}{R}\frac{\partial\zeta}{\partial\phi} \\ \eta &= -\frac{1}{R\cos\phi}\frac{\partial\zeta}{\partial\lambda}. \end{aligned}\right\} \tag{21}$$

The intention of this section is to express these two components in an integral form similar to Neumann's integral. Differentiating Eq. (18) with respect to ϕ and λ, we get

$$\begin{pmatrix} \xi(\phi,\lambda) \\ \eta(\phi,\lambda) \end{pmatrix} = -\frac{1}{4\pi\gamma}\int_0^{2\pi} d\lambda' \int_0^{\pi} \begin{pmatrix} \partial N(R,\psi)/\partial\phi \\ \partial N(R,\psi)/(\cos\phi\partial\lambda) \end{pmatrix} \delta g(\phi',\lambda')\cos\phi' d\phi'. \tag{22}$$

This integral includes differentiations of Neumann's function with respect to ϕ and λ. From the elementary spherical trigonometric identities (Fig. 3) we can readily verify

$$\left.\begin{aligned} \frac{\partial N(R,\psi)}{\partial\phi} &= -\frac{dN(R,\psi)}{d\psi}\cos\alpha \\ \frac{\partial N(R,\psi)}{\partial\lambda} &= -\frac{dN(R,\psi)}{d\psi}\cos\phi\sin\alpha, \end{aligned}\right\} \tag{23}$$

where α is the azimuth reckoned clockwise from the north. Replaced by these relations, Eq. (22) is finally written as

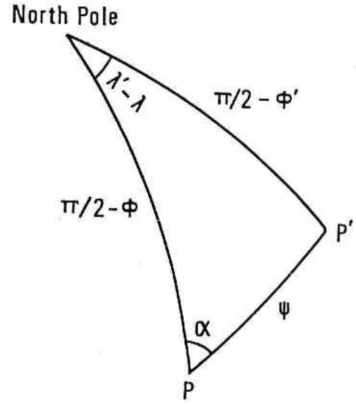

Fig. 3 A spherical triangle showing the spherical coordinates.

$$\begin{pmatrix} \xi(\phi, \lambda) \\ \\ \eta(\phi, \lambda) \end{pmatrix} = \frac{1}{4\pi\gamma}\int_0^{2\pi} d\lambda' \int_0^{\pi} \frac{dN(R, \psi)}{d\psi} \begin{pmatrix} \cos\alpha \\ \\ \sin\alpha \end{pmatrix} \delta g(\phi', \lambda')\cos\phi' d\phi'. \tag{24}$$

This is the modified Vening Meinesz integral to calculate the two components of the deflection of the vertical from the gravity disturbance distribution on the Earth's spherical surface.

Differentiating Neumann's function Eq. (19) with respect to ψ, the modified Vening Meinesz function is obtained as

$$\frac{dN(R, \psi)}{d\psi} = \frac{3}{2}\sin\psi + \operatorname{cosec}\psi\left(1 - \operatorname{cosec}\frac{\psi}{2}\right). \tag{25}$$

The azimuth α is computed from known geodetic positions of P and P′ as

$$\tan\alpha = \frac{\cos\phi'\sin(\lambda' - \lambda)}{\cos\phi\sin\phi' - \sin\phi\cos\phi'\cos(\lambda' - \lambda)} \tag{26}$$

Then we can integrate numerically Eq. (24) with gravity disturbance data measured on the Earth's surface. Special consideration, however, should be made for the numerical integration because $dN(R, \psi)/d\psi \sin\psi$ diverges to infinity when $\psi = 0$. The numerical integration method will be given in the following section.

The direction of the plumbline can be determined by astronomical measurements to obtain the astronomical coordinates of an observation point P. Meanwhile, the GPS measurements also determine its geodetic coordinates. Since these two coordinate systems are independently established, both the coordinates may not coincide with each other. The discrepancies between these two coordinates give two components of the deflection of the vertical. Such obtained deflection is called the "astronomical deflection of the vertical", discriminating from the "gravimetric deflection of the vertical" calculated from the modified Vening Meinesz integration. Although both the deflections of the vertical should be essentially coincident, systematic errors, if they exist, may be caused by a mis—setting of the base ellipsoid. Comparison of both the deflections of the vertical may provide important information regarding base ellipsoid settings.

5. Truncation Error Evaluation

The Neumann and the modified Vening Meinesz integrals are evaluated with summations of the surface elements. In these numerical evaluations the following two points must be considered for lessening computation errors. The first point is that, as pointed out previously, the integral kernels become infinite as $\psi = 0$. The effect of the neighborhood of a computation point P is predominant in both the Neumann and the modified Vening—Meinesz integrals. Therefore, it is necessary for high—accuracy computations to reduce such an effect by modifying the integral formulas. The second point is that the truncation error is caused by neglecting distant integration areas. Our integral formulas involve integrations over the whole surface of the Earth. In practice, however, integrations are extended only over a limited area.

In order to reduce the neighboring effect around P, the Neumann and the modified Vening Meinesz integral formulas are modified using convenient relations:

$$\left.\begin{aligned}&\int_0^{2\pi}d\lambda'\int_0^{\pi}N(R,\psi)\cos\phi' d\phi'=0\\&\int_0^{2\pi}d\lambda'\int_0^{\pi}\frac{dN(R,\psi)}{d\psi}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}\cos\phi' d\phi'=0.\end{aligned}\right\}\tag{27}$$

They are readily proved, as the above integral kernels exclude the zero and first degree terms in the spherical harmonic series. Eq. (27) is multiplied by $\delta g(\phi, \lambda)$ and the products are subtracted from Eq. (18) and (24), respectively. Then we get

$$\left.\begin{aligned}&\zeta(\phi,\lambda)=\frac{R}{4\pi\gamma}\int_0^{2\pi}d\lambda'\int_0^{\pi}N(R,\psi)\{\delta g(\phi',\lambda')-\delta g(\phi,\lambda)\}\cos\phi' d\phi'\\&\begin{pmatrix}\xi(\phi,\lambda)\\ \eta(\phi,\lambda)\end{pmatrix}=\frac{1}{4\pi\gamma}\int_0^{2\pi}d\lambda'\int_0^{\pi}\frac{dN(R,\psi)}{d\psi}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}\{\delta g(\phi',\lambda')-\delta g(\phi,\lambda)\}\cos\phi' d\phi'.\end{aligned}\right\}\tag{28}$$

When $\psi=0$, P coincides with P′, so that $\phi=\phi'$ and $\lambda=\lambda'$. Therefore, the kernels of Eq. (28) become zero as $\psi=0$. This works effectively for reducing the neighboring effect around P.

Next it is necessary to evaluate the truncation effects of neglecting distant integra—tion areas on the computation results of height anomaly and deflection of the vertical. Assuming that the integrations are extended not over the whole surface of the Earth but only up to a spherical distance ψ_0 (see Fig. 4), the truncation errors are given by

$$\left.\begin{aligned}&\delta\zeta(\phi,\lambda)=\frac{R}{4\pi\gamma}\int_0^{2\pi}d\alpha\int_0^{\pi}\bar{N}(R,\psi)\{\delta g(\phi',\lambda')-\delta g(\phi,\lambda)\}\sin\psi d\psi\\&\begin{pmatrix}\delta\xi(\phi,\lambda)\\ \delta\eta(\phi,\lambda)\end{pmatrix}=-\frac{1}{4\pi\gamma}\int_0^{2\pi}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}d\alpha\int_0^{\pi}\bar{V}(R,\psi)\{\delta g(\phi',\lambda')-\delta g(\phi,\lambda)\}\sin\psi d\psi,\end{aligned}\right\}\tag{29}$$

where $\cos\phi' d\phi' d\lambda'$ in Eq. (28) is replaced by $\sin\psi\, d\psi d\alpha$ and newly discontinous functions are defined as:

$$\bar{N}(R,\psi)=\begin{cases}0 & \text{for } 0\leq\psi<\psi_0\\ N(R,\psi) & \text{for } \psi_0\leq\psi\leq\pi\end{cases}\tag{30}$$

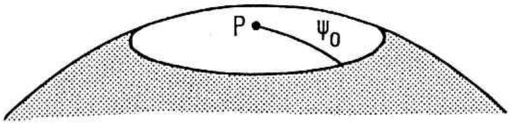

Fig. 4 Truncation area of a spherical cap centering at P with the spherical angle ψ_0.

and

$$\bar{V}(R, \psi)=\begin{cases} 0 & \text{for } 0\leq\psi<\psi_0 \\ -\dfrac{dN(R, \psi)}{d\psi} & \text{for } \psi_0\leq\psi\leq\pi. \end{cases} \tag{31}$$

Mathematical treatments of an integral extended over the whole spherical surface are in most cases much easier than those of an indefinite integral, particularly if the kernels are expressed in forms of spherical surface harmonics. For determining the truncated functions Eq. (30) and (31), they are expanded into a series of Legendre polynomials :

$$\left.\begin{aligned} \bar{N}(R, \psi)&=\sum_{n=0}^{\infty}\frac{2n+1}{2}Q_n P_n(\cos\psi) \\ \bar{V}(R, \psi)&=\sum_{n=0}^{\infty}\frac{2n+1}{2}q_n P_n^1(\cos\psi), \end{aligned}\right\} \tag{32}$$

where Q_n and q_n are truncation coefficients to be determined. The orthogonality conditions of the Legendre and its associated functions are applied to Eq. (32), and then we get

$$\begin{aligned} Q_n&=\int_0^{\pi}\bar{N}(R, \psi)P_n(\cos\psi)\sin\psi d\psi \\ &=\int_{\psi_0}^{\pi}N(R, \psi)P_n(\cos\psi)\sin\psi d\psi, \end{aligned} \tag{33}$$

and similarly

$$q_n=-\frac{1}{n+1}\int_{\psi_0}^{\pi}\frac{dN(R, \psi)}{d\psi}P_n^1(\cos\psi)\sin\psi d\psi. \tag{34}$$

What is the relation between both the truncation coefficients Q_n and q_n? Partially integrating Eq. (34) and using the recurrence formula of the Legendre polynomials :

$$\sin\psi\frac{dP_n^1}{d\psi}+P_n^1(\cos\psi)\cos\psi=n(n+1)P_n(\cos\psi)\sin\psi, \tag{35}$$

for $n\geqq1$ we obtain

$$\begin{aligned} q_n&=-\frac{1}{n(n+1)}[N(R, \psi)P_n^1(\cos\psi)\sin\psi]_{\psi_0}^{\pi} \\ &\quad+\frac{1}{n(n+1)}\int_{\psi_0}^{\pi}N(R, \psi)\frac{d}{d\psi}\{P_n^1(\cos\psi)\sin\psi\}d\psi \\ &=Q_n+\frac{1}{n(n+1)}N(R, \psi_0)P_n^1(\cos\psi_0)\sin\psi_0. \end{aligned} \tag{36}$$

This gives the relation between Q_n and q_n. Q_n can be computed from q_n and vice versa.

Hagiwara (1972) first derived a similar relation regarding truncation coefficients of the Stokes and the Vening—Meinesz integrals.

Eq. (34) is analytically obtained by the use of an expansion series:

$$P_n^1(\cos\psi)=\cos\frac{\psi}{2}\sum_{m=0}^{n-1}\frac{(-1)^m(n+m+1)!}{m!(m+1)!(n-m-1)!}\sin^{2m+1}\frac{\psi}{2}. \tag{37}$$

Substituting Eq. (37) into Eq. (34) and taking Eq. (25) into account, for $n \geqq 1$ we obtain

$$q_n=-\frac{1}{n(n+1)}\sum_{m=1}^{n}\frac{(-1)^m(n+m)!}{(m!)^2(n-m)!}\left[\frac{1}{2m-1}\{1-2mt^{2m-1}+(2m-1)t^{2m}\}\right.$$

$$\left.-\frac{6m}{(m+1)(m+2)}\{1-(m+2)t^{2m+2}+(m+1)t^{2m+4}\}\right], \tag{38}$$

where $t=\sin(\psi_0/2)$. Q_n is computed by Eq. (36) from the obtained value of q_n for cases where $n \geqq 1$. In the special case of $n=0$, the direct integration of Eq. (33) becomes

$$Q_0=-t(1-t)(2-3t-3t^2)-2t^2\log t-2(1-t^2)\log(1+t). \tag{39}$$

However, q_0 is not defined. Fig. 5 shows the behaviors of low—degree terms of Q_n and q_n. Round—off computation errors may increase cumulatively in the summation of the power series up to a large number of n. Special consideration must be taken for reducing computation errors in the computer program used.

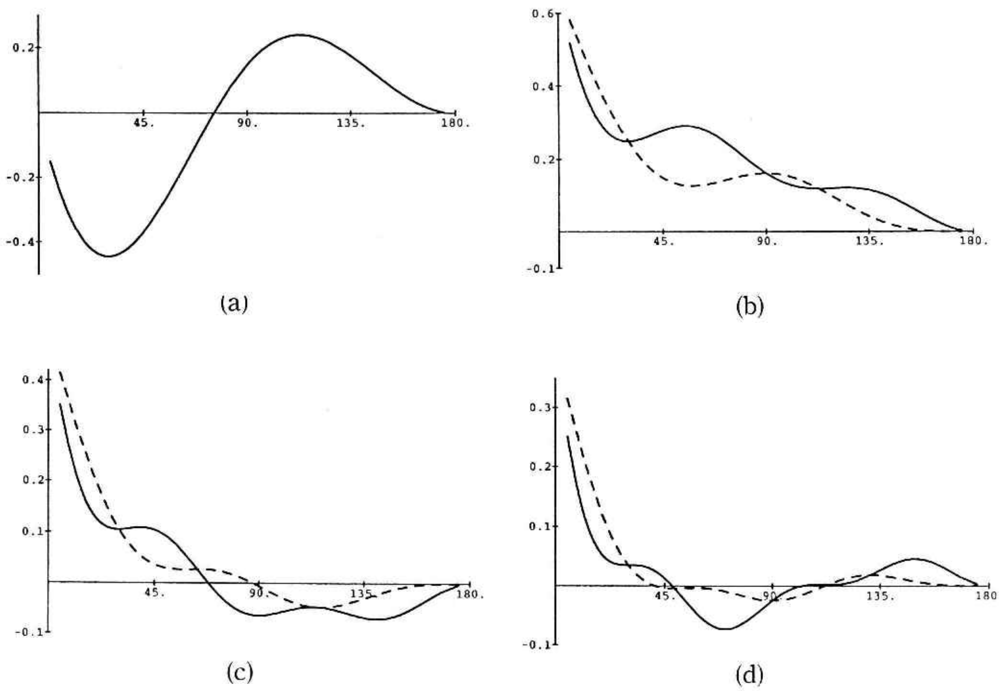

Fig. 5 Truncation error coefficients of the Neumann integral (solid line) and the modified Vening Meinesz integral (dotted line). (a) Q_0, (b) Q_2 and q_2, (c) Q_3 and q_3, and (d) Q_4 and q_4.

Finally, inserting Eq. (32) into Eq. (29) and applying the orthogonality conditions of the spherical harmonic functions, the truncation errors expressed in spherical harmonic expansion series are obtained :

$$\left.\begin{aligned} \delta\zeta(\phi,\lambda) &= \frac{R}{2\gamma}\left\{\sum_{n=2}^{\infty} Q_n \delta g_n(\phi,\lambda) - Q_0 \delta g(\phi,\lambda)\right\} \\ \begin{pmatrix} \delta\xi(\phi,\lambda) \\ \delta\eta(\phi,\lambda) \end{pmatrix} &= -\frac{1}{2\gamma}\sum_{n=2}^{\infty} q_n \begin{pmatrix} \partial\delta g_n(\phi,\lambda)/\partial\phi \\ \partial\delta g_n(\phi,\lambda)/(\cos\phi\partial\lambda) \end{pmatrix}. \end{aligned}\right\} \quad (40)$$

The spherical harmonic expansion coefficients C_n^m and S_n^m of the geopotential, which are derived from perturbation analyses of satellite orbit elements, can be used for computing $\delta g_n(\phi, \lambda)$. The first formula of Eq. (40) indicates that the truncation error of the height anomaly is evaluated by summing up the thus far obtained $\delta g_n(\phi, \lambda)$ with coefficients Q_n. Similarly the truncation errors of the two components of deflection of the vertical are evaluated by the second formula of Eq. (40). The mathematical procedure for deriving these formulas will not be introduced herein, because it is essentially similar to one for deriving the truncation error formulas of gravity anomaly. For this detailed derivation procedure refer to Hagiwara (1976).

6. Inverse Problem

The GPS satellite distance—measuring technique makes it possible to determine the ocean surface height above the ellipsoid. The static water surface forms an equipotential surface. Although the real ocean surface is disturbed with currents, waves, atmospheric pressure and temperature, etc., its topography can be approximated to the height anomaly on the average. This fact indicates that the GPS technique can determine directly the height anomaly in ocean areas. Independently of this technique, the height anomaly can also be calculated from gravity disturbance data measured on the ocean surface. If some systematic differences are found between these two height anomalies, they may provide important information regarding the settings of the ellipsoid to the Earth's coordinates.

The inverse formula of the Stokes integral formula was first derived only out of theoretical interest, but later it was actually applied in predicting approximately gravity anomaly from the ocean surface topography measured in ocean areas where no gravity measurements had been made. At that time many unobserved areas in a large part of the Antarctic Ocean remained, but nowadays these gaps have been gradually filled with sea—surface gravimetries. Notwithstanding, it will be of advantage to the GPS—gravity combination surveys in ocean areas to derive a similar inverse formula of the Neumann integral. The inverse Neumann formula involves an integration over the whole surface of the Earth. The purpose of this section is to derive the inverse Neumann formula as well as to formulate trucation errors resulting from the neglect of remote integration ranges.

To begin with, we put $r=R$ in Eq. (10) and divide both sides by γ, and by taking Bruns' formula into consideration, we express the height anomaly in the spherical surface harmonic expansion series:

$$\zeta(\phi, \lambda)=\frac{R}{\gamma}\sum_{n=2}^{\infty}\frac{\delta g_n(\phi, \lambda)}{n+1}. \tag{41}$$

If ζ is expanded in a series of spherical surface harmonic functions of ζ_n, from Eq. (41) we obtain

$$\delta g_n(\phi, \lambda)=\frac{\gamma(n+1)}{R}\zeta_n(\phi, \lambda). \tag{42}$$

The summation of both sides of Eq. (42) with respect to $n \geqq 2$ gives

$$\delta g(\phi, \lambda)=\frac{\gamma}{R}\left\{\zeta(\phi, \lambda)+\sum_{n=2}^{\infty}n\zeta_n(\phi, \lambda)\right\}. \tag{43}$$

Heiskanen and Moritz (1967) introduced a relation holding for an arbitrary function F defined on the surface of a sphere, that is

$$\frac{R^2}{2\pi}\int_0^{2\pi}d\alpha\int_0^{\pi}\frac{F(\phi', \lambda')-F(\phi, \lambda)}{\ell_0^3}\sin\psi d\psi=-\frac{1}{R}\sum_{n=0}^{\infty}nF_n(\phi, \lambda), \tag{44}$$

where $\ell_0=2R\sin(\psi/2)$. The last term of the righthand side of Eq. (43) has a similar form to the righthand side of Eq. (44). The summation starts from $n=0$ in Eq. (44), but ζ_n has no terms of $n=0$ and 1. Replacing F by ζ, Eq. (43) is rewritten in a new form:

$$\delta g(\phi, \lambda)=\frac{\gamma}{R}\left\{\zeta(\phi, \lambda)-\frac{1}{16\pi}\int_0^{2\pi}d\alpha\int_0^{\pi}\frac{\zeta(\phi', \lambda')-\zeta(\phi, \lambda)}{\sin^3(\psi/2)}\sin\psi d\psi\right\}. \tag{45}$$

This is the inverse formula of the Neumann integral formula Eq. (18) for evaluating δg from the known distribution of ζ on the Earth's surface.

Eq. (45) involves a two—dimensional integration extended over the whole surface of the Earth. Practically the integration is extended only over a limited area because the kernal function seems to converge rapidly to zero with the distance from P. If the integration range is limited to a spherical cap centered at P with a spherical distance ψ_0, Eq. (45) is expressed by an integral formula in which the integration range $0 \leqq \psi \leqq \pi$ is replaced by $0 \leqq \psi \leqq \psi_0$. The corresponding truncation error is expressed as

$$\varepsilon(\phi, \lambda)=-\frac{\gamma}{2\pi R}\int_0^{2\pi}d\alpha\int_0^{\pi}K(\psi)\{\zeta(\phi', \lambda')-\zeta(\phi, \lambda)\}\sin\psi d\psi, \tag{46}$$

with a discontinuous function:

$$K(\psi)=\begin{cases} 0 & \text{for } 0\leq\psi\leq\psi_0 \\ \dfrac{1}{8\sin^3(\psi/2)} & \text{for } \psi_0\leq\psi\leq\pi. \end{cases} \tag{47}$$

In order to express ε in a series of spherical surface harmonics similarly to Eq. (32), the function $K(\psi)$ is expanded into a series of the Legendre polynomials:

$$K(\psi)=\sum_{n=0}^{\infty}\frac{2n+1}{2}K_nP_n(\cos\psi), \tag{48}$$

where by the orthogonality relation of the Legendre polynomials the coefficient K_n is written as

$$K_n=\int_0^{\pi}K(\psi)P_n(\cos\psi)\sin\psi d\psi$$

$$=\frac{1}{8}\int_{\psi_0}^{\pi}\frac{P_n(\cos\psi)}{\sin^3(\psi/2)}\sin\psi d\psi. \tag{49}$$

This equation indicates that K_n can be determined as a function of the truncation angle ψ_0. Substituting Eq. (48) into Eq. (46) and taking the orthogonality relation of the Legendre polynomials into consideration, we obtain

$$\varepsilon(\phi,\lambda)=\frac{\gamma}{R}\left\{K_0\zeta(\theta,\lambda)-\sum_{n=2}^{\infty}K_n\zeta_n(\phi,\lambda)\right\}. \tag{50}$$

It is noticed that this equation has a similar form of the first equation of Eq. (40).

Next the truncation error coefficient K_n is formulated by solving the integral Eq. (49). This integral is quite similar to an integral which appears in the derivation process of the Stokes integral truncation error (Hagiwara, 1976). The zero and first degree integrations are easily performed as

$$\left.\begin{aligned}K_0&=-\frac{1}{2}\left(1-\mathrm{cosec}\frac{\psi_0}{2}\right)\\K_1&=K_0-\left(1-\sin\frac{\psi_0}{2}\right)\end{aligned}\right\}. \tag{51}$$

The general degree terms of K_n can not be expressed by elementary functions. A complicated integral, including the Legendre polynomials, sometimes takes on a much simpler recursive form. Hagiwara (1976) proposed a method for computing K_n by using the following recursive formula :

$$K_{m+1}-2K_m+K_{m-1}=-\frac{P_{m+1}(\cos\psi_0)-P_{m-1}(\cos\psi_0)}{2(2m+1)\sin(\psi_0/2)}. \tag{52}$$

Summing up both sides of Eq. (52), weighted by $n-m$ from $m=1$ to $n-1$, we obtain the general expression of K_n :

$$K_n=K_0-n\left(1-\sin\frac{\psi_0}{2}\right)$$

$$-\frac{1}{2\sin(\psi_0/2)}\sum_{m=1}^{n-1}\frac{n-m}{2m+1}\{P_{m+1}(\cos\psi_0)-P_{m-1}(\cos\psi_0)\}. \tag{53}$$

Theoretically K_n can be computed by Eq. (53) even for a large number of n. In practice, however, Eq. (53) creates an invalid solution for a large number of n due to round-off errors cummlative in the summation of the Legendre polynomials. For a high degree term of K_n Saito (1980) derived an asymptotic formula given by

$$K_n \simeq -\frac{\sin\{(n+1/2)\psi_0-\pi/4\}}{8(n+1)\sin^3(\psi_0/2)}\sqrt{\frac{2\sin\psi_0}{n\pi}}. \tag{54}$$

This formula indicates that K_n is in the order of $n^{-3/2}$ and decreases to zero when n becomes infinite.

It is a well known fact that, in cases of the Fourier analysis in the Cartesian coordinates, the truncation of a function causes the Gibbs oscillating phenomenon. A window function is sometimes used for reducing such a phenomenon. The truncation of a spherical surface harmonic function behaves similarly to the Fourier case. The Gibbs phenomenon in the spherical coordinates is a subject to be studied later to search for a window function fitting to the spherical surface harmonic analysis.

7. Molodenskii Approach

Molodenskii et al. (1962) introduced the modern approach to physical geodesy. In the conservative approach one must know the density distribution of the Earth's interior materials above the ellipsoid or make assumptions concerning it. The basic importance of the Molodenskii approach was to prove that the physical surface of the Earth could be determined from geodetic measurements alone without knowledge of the Earth's density. This requires abandoning the ambiguous concept of the geoid. The technical term of "height anomaly" was first used in their theory. It is very difficult, however, to understand completely the Molodenskii theory because the mathematical formulation is not only abstract but complicated. Heiskanen and Moritz (1967) proposed an approximation method of the Molodenskii theory by using the vertical derivative of the gravity anomaly. Since then the Molodenskii theory has become more acceptable owing to Heiskanen and Moritz's excellent interpretations.

Molodenskii et al. (1962) gave an ellegant solution for the geodetic boundary—value problem of the Earth's gravity potential field in a form of series $G_n(n=1, 2, \cdots)$. They showed that the first term G_1 is the conventional Stokes integral of gravity anomaly Δg, but the higher terms are terrain corrections for the undulation of the Earth's surface. The Molodenskii theory shows that the terrain correction is not related to the density of the Earth's materials. Taking only the first term into consideration, Heiskanen and Moritz (1967) expressed the potential disturbance on the Earth's surface as the Stokes integral of $\Delta g+G_1$ instead of Δg.

On the analogy of the G_1 term in the Stokes integral, the Molodenskii approach can be extended to the Neumann boundary—value problem for GPS—gravity combined geodetic measurements. The gravity disturbance δg is actually measured on the undulated Earth's surface. Now a fictitious field of gravity disturbance δg^* is assumed on the ellipsoid as shown in Fig. 6. If one takes the spherical approximation, Eq. (18) is rewritten here again with the new notation :

$$\zeta^*(\phi, \lambda)=\frac{R}{4\pi\gamma}\int_0^{2\pi}d\lambda'\int_0^{\pi}N(R, \psi)\delta g^*(\phi', \lambda')\cos\phi' d\phi', \tag{55}$$

where ζ^* is the fictitious height anomaly on the ellipsoid. However, a relation similar to Eq. (55) does not hold between δg and ζ, because the Earth's surface is not assumed to be spherical but undulated.

Analogically to the Molodenskii G_1 term in the Stokes integral formula, another integral formula is defined:

$$\zeta(\phi, \lambda)=\frac{R}{4\pi\gamma}\int_0^{2\pi}d\lambda'\int_0^{\pi}N(R, \psi)\{\delta g(\phi', \lambda')+G_1(\phi', \lambda')\}\cos\phi' d\phi'. \tag{56}$$

The above G_1 term is different from the original Molodenskii term, but it is newly defined with gravity disturbance as

$$G_1(\phi, \lambda)=\frac{R^2}{2\pi}\int_0^{2\pi}d\lambda'\int_0^{\pi}\frac{h(\phi', \lambda')-h(\phi, \lambda)}{\ell_0^3}\delta g(\phi', \lambda')\cos\phi' d\phi', \tag{57}$$

where h is the true height above the ellipsoid (see Fig. 6), and $\ell_0=2R\sin(\psi/2)$. In the original Molodenskii formula, h and δg in Eq. (57) are replaced respectively by the normal height H and the gravity anomaly Δg.

On the other hand, Heiskanen and Moritz (1967) started their theory from the vertical gradient of gravity anomaly. They assumed that the fictitious gravity anomaly Δg^* is approximated by an additional term concerning the vertical gradient of Δg. A similar approximation can be made to the relation between δg and δg^*, that is

$$\delta g^*=\delta g-h\left[\frac{\partial\delta g}{\partial r}\right]_{r=R}. \tag{58}$$

According to their theory, the vertical derivative of Δg is expressed as an integral formula on the assumption that $H\,\Delta g/R$ is a very small quantity. We can lead a similar integral formula for our gravity disturbance:

$$\frac{\partial\delta g}{\partial r}=\frac{R^2}{2\pi}\int_0^{2\pi}d\lambda'\int_0^{\pi}\frac{\delta g(\phi', \lambda')-\delta g(\phi, \lambda)}{\ell_0^3}\cos\phi' d\phi'. \tag{59}$$

Eq. (58) is rewritten with a new notation G_{11}, which was originally used by Heiskanen and Moritz (1967), such as

$$\delta g^*=\delta g+G_{11}. \tag{60}$$

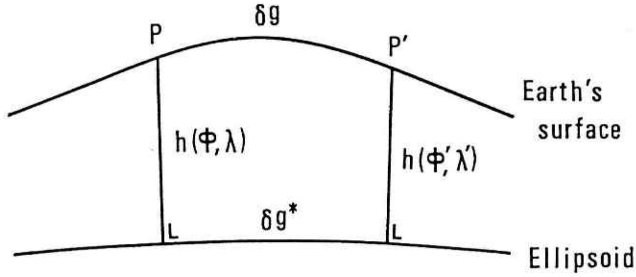

Fig. 6 The gravity disturbance, δg, is measured on the undulated Earth's surface. The ficitious gravity disturbance, δg^*, can be calculated from δg.

Comparison between Eq. (58) and (60) gives $G_{11}=-h[\partial\delta g/\partial r]_{r=R}$, then from Eq. (59) we get

$$G_{11}(\phi,\lambda)=-\frac{R^2}{2\pi}h(\phi,\lambda)\int_0^{2\pi}d\lambda'\int_0^{\pi}\frac{\delta g(\phi',\lambda')-\delta g(\phi,\lambda)}{\ell_0^3}\cos\phi' d\phi'. \tag{61}$$

Substituting Eq. (60) into Eq. (55), we obtain

$$\zeta^*(\phi,\lambda)=\frac{R}{4\pi\gamma}\int_0^{2\pi}d\lambda'\int_0^{\pi}N(R,\psi)\{\delta g(\phi',\lambda')+G_{11}(\phi',\lambda')\}\cos\phi' d\phi'. \tag{62}$$

The geodetical importance of this formula is that one can compute the fictitious height anomaly on the ellipsoid ("quasigeoid" by Molodenskii et al., 1962) from the height anomaly actually measured on the Earth's surface. The G_{11} term eliminates the effect of terrain undulation on the height anomaly.

In the same way as Eq. (58) holding for gravity disturbances, the fictitious height anomaly ζ^* can be written with its vertical derivative as

$$\zeta^*=\zeta-h\left[\frac{\partial\zeta}{\partial r}\right]_{r=R}. \tag{63}$$

If the additional term is denoted by $F=-h[\partial\zeta/\partial r]_{r=R}$ and a contribution of F to the gravity disturbance field by G_{12}, F can be written in a way similar to Eq. (62) as

$$F(\phi,\lambda)=-\frac{R}{4\pi\gamma}\int_0^{2\pi}d\lambda'\int_0^{\pi}N(R,\psi)G_{12}(\phi',\lambda')\cos\phi' d\phi'. \tag{64}$$

Substituting $\zeta^*=\zeta+F$ into the lefthand side of Eq. (62), we get

$$\zeta(\phi,\lambda)=\frac{R}{4\pi\gamma}\int_0^{2\pi}d\lambda'\int_0^{\pi}N(R,\psi)\{\delta g(\phi',\lambda')+G_{11}(\phi',\lambda')+G_{12}(\phi',\lambda')\}\cos\phi' d\phi'. \tag{65}$$

Comparing Eq. (65) with Eq. (56), one finds $G_1=G_{11}+G_{12}$.

Eq. (45) is the inverse formula of Eq. (18). In a similar way, the inverse formula of Eq. (64) can be obtained in the form :

$$G_{12}(\phi,\lambda)=\frac{\gamma}{R}\left\{F(\phi,\lambda)-\frac{1}{2\pi}\int_0^{2\pi}d\lambda'\int_0^{\pi}\frac{F(\phi',\lambda')-F(\phi,\lambda)}{\ell_0^3}\cos\phi' d\phi'\right\}, \tag{66}$$

where $\sin\psi\, d\psi d\alpha$ is replaced by $\cos\phi' d\phi' d\lambda$.

The physical importance of Eq. (65) is that, if we divide the Molodenskii G_1 into two parts, the first part, G_{11}, is a correction term to δg considering the vertical derivative of gravity disturbance and the second part, G_{12}, is a contribution of height anomaly differences between ζ and ζ^* to the gravity disturbance field.

The fictitious components of deflection of the vertical, ξ^* and η^*, on the ellipsoid can be expressed in a similar way as shown above. The modified Vening—Meinesz integral formula of gravity disturbance on the ellipsoid is written as

$$\begin{pmatrix}\xi^*(\phi,\lambda)\\ \eta^*(\phi,\lambda)\end{pmatrix}=\frac{1}{4\pi\gamma}\int_0^{2\pi}d\lambda'\int_0^{\pi}\frac{dN(R,\psi)}{d\psi}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}\{\delta g(\phi',\lambda')+G_{11}(\phi',\lambda')\}\cos\phi' d\phi'. \tag{67}$$

This integral formula corresponds to Eq. (62). The downward continuation of the two components are approximately given by,

$$\begin{pmatrix} \xi^* \\ \eta^* \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} - h \begin{pmatrix} [\partial\xi/\partial r]_{r=R} \\ [\partial\eta/\partial r)_{r=R} \end{pmatrix}. \tag{68}$$

The second terms of the righthand side of Eq. (68) are not observable quantities, so that they are reformed by using

$$\frac{\partial\zeta}{\partial r} = \frac{2}{R}\zeta - \frac{\delta g}{\gamma}, \tag{69}$$

which is obtained from differentiating the Bruns formula with respect to r. Here the horizontal derivatives of γ are neglected because they are very small. Differentiating again both sides of Eq. (69) with respect to ϕ and λ, and taking Eq. (21) into account, we have

$$\left.\begin{aligned} \left[\frac{\partial\xi}{\partial r}\right]_{r=R} &= \frac{2}{R}\xi + \frac{1}{\gamma R}\frac{\partial\delta g}{\partial\phi} \\ \left[\frac{\partial\eta}{\partial r}\right]_{r=R} &= \frac{2}{R}\eta + \frac{1}{\gamma R}\frac{\partial\delta g}{\cos\phi\partial\lambda} \end{aligned}\right\}. \tag{70}$$

Substituting Eq. (70) into Eq. (68), we get

$$\begin{pmatrix} \xi^* \\ \eta^* \end{pmatrix} = \left(1 - \frac{2h}{R}\right)\begin{pmatrix} \xi \\ \eta \end{pmatrix} - \frac{h}{\gamma R}\begin{pmatrix} \partial\delta g/\partial\phi \\ \partial\delta g/(\cos\phi\partial\lambda) \end{pmatrix}. \tag{71}$$

Eliminating ξ^* and η^* in both Eq. (67) and (71) and neglecting $2h/R$ in Eq. (71), we obtain the final expression of the integral formula :

$$\begin{pmatrix} \xi(\phi,\lambda) \\ \eta(\phi,\lambda) \end{pmatrix} = \frac{1}{4\pi\gamma}\int_0^{2\pi} d\lambda' \int_0^{\pi} \frac{dN(R,\psi)}{d\psi} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \{\delta g(\phi,\lambda') + G_{11}(\phi',\lambda')\} \cos\phi' d\phi'$$

$$+ \frac{h(\phi,\lambda)}{\gamma R}\begin{pmatrix} \partial\delta g(\phi,\lambda)/\partial\phi \\ \partial\delta g(\phi,\lambda)/(\cos\phi\partial\lambda) \end{pmatrix}. \tag{72}$$

This integral formula has an additional term regarding the horizontal derivatives of gravity disturbance. In the original Molodenskii approach the horizontal derivatives of height anomaly appear instead of those of gravity disturbance. In practice, however, the numerical computations of those of gravity disturbance are much easier because those of height anomaly attain quite large values in steep terrains. Heiskanen and Moritz (1967) proposed a more preferable formula including vertical derivatives of gravity anomaly. I think that an equivalent method may possibly be applied to the Neumann boundary

—value problem.

Recently the Molodenskii theory has been reestimated by the use of the two—dimensional fast Fourier transform (FFT) algorithm in the Cartesian coordinates (Sideris, 1985; Sideris and Schwarz, 1986, 1988). The integral solutions of the height anomaly and the deflections of the vertical are reformulated to obtain convolution integrals in planar approximation. The local contributions to the Vening—Meinesz integral are efficiently evaluated at rectangular grid points of gravity anomaly data array (Zavattero, 1987). The spectral evaluation of the Molodenskii integrals in the frequency domain and the related numerical computation method are reviewed in detail by Schwarz et al. (1990). The methods presented in these papers can be used for the Neumann problem with gravity disturbance.

8. Fourier Transform Method

The flat—earth approximation neglecting the curvature of the Earth's surface can be made for local gravity field calculations at rectangular grid points of the data in the Cartesian coordinates. It has widely been recognized that such an approximation method provides extremely efficient computations of integral formulas of physical geodesy, such as the Vening—Meinesz and the Molodenskii integrals. The advantage of this method is that most of these integral formulas can be expressed in the form of a two—dimensional convolution integral which is efficiently evaluated by the Fourier transform method in the frequency domain.

In recent years many papers have been published formulating convolution integrals of flat—earth physical geodesy (e.g. Forsberg, 1985; Vassiliou, 1988; Sideris and Schwarz, 1988; Gleason, 1990; Schwarz et al., 1990). The availability of gridded gravity and elevation data has resulted in the successful use of the 2—D Fourier transform method for computing spatial deflections of the vertical by the Vening—Meinesz convolution integral. When a great amount of data is available in gridded form, the use of the FFT technique is clearly appropriate. The Molodenskii series solution is one of the most interesting targets for physical geodesists to reformulate in a convolution form and to use to evaluate the terrain correction terms with a combination of heights and gravity anomalies. The original Molodenskii solution is simplified starting from the analytical continuation to point level solution (Moritz, 1980).

In this section the modified Vening—Meinesz and Molodenskii integrals with gravity disturbance in planar approximation are examined. A rectangular coordinate system(x, y, z) is introduced, where the x, y and z—axis are pointing north, east and vertically downward. The ground surface lies at $z=0$. The coordinates of an observation point P and of a calculation point P$'$ are (x, y, z) and (x$'$, y$'$, z$'$). By denoting wavenumbers (angular frequencies) u and v with respect to the x and y—axis, we define the 2—D Fourier transform of an arbitrary space—domain function $f(x, y)$ as

$$\tilde{f}(u, v)=\iint_{-\infty}^{\infty} f(x, y)\exp\{-i(ux+vy)\}dx, dy, \tag{73}$$

where i is the imaginary unit ($i=\sqrt{-1}$). Inversely the function $f(x, y)$ can be obtained from its frequency—domain function $\tilde{f}(u, v)$ by means of the inverse 2—D Fourier transform :

$$f(x, y)=\frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \tilde{f}(u, v)\exp\{i(ux+vy)\}du\,dv. \tag{74}$$

The interest is in formulating geodetic convolution integrals. By denoting two arbitrary space—domain functions as $g(x, y)$ and $h(x, y)$, the following convolution integral is defined as :

$$f(x, y)=\iint_{-\infty}^{\infty} g(x-x', y-y')h(x', y')dx'dy'. \tag{75}$$

The frequency—domain expression of Eq. (75) has a very simple form :

$$\tilde{f}(u, v)=\tilde{g}(u, v)\tilde{h}(u, v), \tag{76}$$

where $\tilde{g}$ and $\tilde{h}$ are the 2—D Fourier transforms of g and h. Eq. (76) is readily derived from Eq. (75). The proof can be found in textbooks of the Fourier transform.

On the basis of the mathematical background introduced here, most integral formulas in spherical—earth physical geodesy can be reformulated. To begin with, the upward continuation of gravity disturbance is discussed, which is assumed to be caused by the spatial density distribution ρ inside the half—space ($z\leq 0$). Allowing the distance between two points P and P′ to be

$$\ell=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}, \tag{77}$$

we can write the potential disturbance as

$$T(x, y, z)=G\int_{-\infty}^{0}dz'\iint_{-\infty}^{\infty}\frac{\rho(x', y', z')}{\ell}dx'dy' \tag{78}$$

(see Fig. 7). Differentiating T with respect to z, gravity disturbance is thus obtained :

$$\delta g(x, y, z)=G\int_{-\infty}^{0}(z'-z)dz'\iint_{-\infty}^{\infty}\frac{\rho(x', y', z')}{\ell^3}dx'dy'. \tag{79}$$

It is seen that both integrals of Eq. (78) and (79) have forms similar to that of Eq. (75), h corresponding to ρ and g to $1/\ell$ in Eq. (78) and $1/\ell^3$ in Eq. (79). That is to say, Eq. (78) and (79) are convolution integrals, which have forms similar to Eq. (76) in the frequency domain. The 2—D Fourier transforms of $1/\ell$ and $1/\ell^3$ become respectively

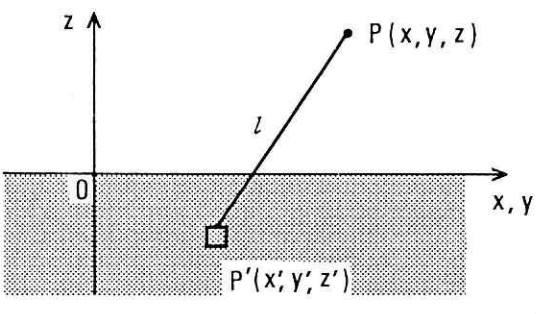

Fig. 7 Cartesian coordinates of two points P and P′.

$$\iint_{-\infty}^{\infty} \frac{\exp\{-i(ux+vy)\}}{\sqrt{x^2+y^2+(z-z')^2}} dx\, dy = \frac{2\pi \exp(-|z-z'|\sqrt{u^2+v^2})}{\sqrt{u^2+v^2}} \tag{80}$$

and

$$\iint_{-\infty}^{\infty} \frac{\exp\{-i(ux+vy)\}}{\{x^2+y^2+(z-z')^2\}^{3/2}} dx\, dy = \frac{2\pi \exp(-|z-z'|\sqrt{u^2+v^2})}{|z-z'|}. \tag{81}$$

Thus the 2–D Fourier transforms of Eq. (78) and (79) can be noted as

$$\tilde{T}(u, v, z) = \frac{2\pi G \tilde{I}(u, v) \exp(-z\sqrt{u^2+v^2})}{\sqrt{u^2+v^2}} \tag{82}$$

and

$$\delta \tilde{g}(u, v, z) = 2\pi G \tilde{I}(u, v) \exp(-z\sqrt{u^2+v^2}), \tag{83}$$

where we define

$$\tilde{I}(u, v) = \int_{-\infty}^{0} \tilde{\rho}(u, v, z') \exp(z'\sqrt{u^2+v^2}) dz'. \tag{84}$$

Now assuming that $z=0$ in Eq. (82) and (83), we get

$$\tilde{T}(u, v, 0) = \frac{2\pi G \tilde{I}(u, v)}{\sqrt{u^2+v^2}} \tag{85}$$

and

$$\delta \tilde{g}(u, v, 0) = 2\pi G \tilde{I}(u, v). \tag{86}$$

Eliminating $\tilde{I}$ in Eq. (82) by Eq. (85) and $\tilde{I}$ in Eq. (83) by Eq. (86), very important relations are obtained :

$$\left.\begin{aligned} \tilde{T}(u, v, z) &= \tilde{T}(u, v, 0) \exp(-z\sqrt{u^2+v^2}) \\ \delta \tilde{g}(u, v, z) &= \delta \tilde{g}(u, v, 0) \exp(-z\sqrt{u^2+v^2}) \end{aligned}\right\} \tag{87}$$

These formulas have forms similar to Eq. (76). Referring to Eq. (81), the inverse 2–D Fourier transforms of Eq. (87) can then be derived as

$$\left.\begin{aligned} T(x, y, z) &= \frac{z}{2\pi} \iint_{-\infty}^{\infty} \frac{T(x', y', 0)}{\{(x-x')^2+(y-y')^2+z^2\}^{3/2}} dx' dy' \\ \delta g(x, y, z) &= \frac{z}{2\pi} \iint_{-\infty}^{\infty} \frac{\delta g(x', y', 0)}{\{(x-x')^2+(y-y')^2+z^2\}^{3/2}} dx' dy' \end{aligned}\right\} \tag{88}$$

These formulas give a very important "upward continuation" relation between values at the ground surface and at a certain elevation. If the surface values of $T(x, y, 0)$ are known, one can calculate the space values of $T(x, y, z)$ by means of the first formula of Eq. (88). Similarly one can calculate $\delta g(x, y, z)$ from $\delta g(x, y, 0)$. It is noticed that T and δg satisfy the Laplace equation Eq. (3), and that these two formulas in Eq. (88)

are solutions of the Dirichlet boundary—value problem in the Cartesian coordinates.

The kernels of the integrals in Eq. (88) become $1/(2\pi z^2)$ when P coincides with P', i.e. $x=x'$ and $y=y'$. If z is very small, the kernels diverge to infinity in that case. In order to avoid such a computational difficulty, Eq. (88) is modified by using a convenient identity as follows :

$$\frac{z}{2\pi}\iint_{-\infty}^{\infty}\frac{dx'dy'}{\{(x-x')^2+(y-y')^2+z^2\}^{3/2}}=1. \tag{89}$$

By multipling both sides of Eq. (89) by $T(x, y, 0)$ and $\delta g(x, y, 0)$, the two formulas in Eq. (88) can respectively be modified as :

$$\left.\begin{aligned} T(x, y, z)&=T(x, y, 0)+\frac{z}{2\pi}\iint_{-\infty}^{\infty}\frac{T(x', y', 0)-T(x, y, 0)}{\{(x-x')^2+(y-y')^2+z^2\}^{3/2}}dx'dy' \\ \delta g(x, y, z)&=\delta g(x, y, 0)+\frac{z}{2\pi}\iint_{-\infty}^{\infty}\frac{\delta g(x', y', 0)-\delta g(x, y, 0)}{\{(x-x')^2+(y-y')^2+z^2\}^{3/2}}dx'dy' \end{aligned}\right\} . \tag{90}$$

The modifications are similar to those of Eq. (28) by obtaining the height anomaly and two components of deflections of the vertical by means of the Neumann and the modified Vening—Meinesz integrals.

On the other hand, another important relation can be derived by eliminating $\tilde{\Gamma}$ from both Eq. (82) and (86), that is

$$\tilde{T}(u, v, z)=\frac{\exp(-z\sqrt{u^2+v^2})}{\sqrt{u^2+v^2}}\delta\tilde{g}(u, v, 0). \tag{91}$$

Referring to Eq. (80), we obtain the space—domain expression of Eq. (91) as

$$T(x, y, z)=\frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{\delta g(x', y', 0)}{\sqrt{(x-x')^2+(y-y')^2+z^2}}dx'dy'. \tag{92}$$

This corresponds to the Neumann integral Eq. (13), obtaining potential disturbance in space from the known surface distribution of gravity disturbance. When ψ is very small, $N(r, \psi)\fallingdotseq 2R/\ell$ as in Eq. (16). Then Eq. (13) is approximated by Eq. (92) in planar approximation. By dividing both the sides of Eq. (92) by the normal gravity γ at $z=0$, the height anomaly is obtained through the Bruns formula as

$$\zeta(x, y)=\frac{1}{2\pi\gamma}\iint_{-\infty}^{\infty}\frac{\delta g(x', y', 0)}{\sqrt{(x-x')^2+(y-y')^2}}dx'dy'. \tag{93}$$

This formula corresponds to Eq. (18) for a spherical Earth.

Next we consider two components of diflections of the vertical for the flat—earth approximation. In the present case they are given by

$$\left.\begin{aligned} \xi&=-\frac{\partial\zeta}{\partial x} \\ \eta&=-\frac{\partial\zeta}{\partial y} \end{aligned}\right\} . \tag{94}$$

Differentiating Eq. (93) with respect to x and y, we obtain

$$\begin{pmatrix} \xi(x, y) \\ \\ \eta(x, y) \end{pmatrix} = \frac{1}{2\pi\gamma} \iint_{-\infty}^{\infty} \frac{\delta g(x', y', 0)}{\{(x-x')^2+(y-y')^2\}^{3/2}} \begin{pmatrix} x-x' \\ \\ y-y' \end{pmatrix} dx'dy'. \tag{95}$$

The integral kernel diverges infinitely when P coincides with P', i.e. $x=x'$ and $y=y'$. Such a computational difficulty can be overcome by the following procedure. Using an identity

$$\iint_{-\infty}^{\infty} \frac{1}{\{(x-x')^2+(y-y')^2\}^{3/2}} \begin{pmatrix} x-x' \\ \\ y-y' \end{pmatrix} dx'dy' = 0, \tag{96}$$

multiplied by $\delta g(x, y, 0)$ and subtracted the product from Eq. (95), we obtain a modified form of Eq. (95).

$$\begin{pmatrix} \xi(x, y) \\ \\ \eta(x, y) \end{pmatrix} = \frac{1}{2\pi\gamma} \iint_{-\infty}^{\infty} \frac{\delta g(x', y', 0) - \delta g(x, y, 0)}{\{(x-x')^2+(y-y')^2\}^{3/2}} \begin{pmatrix} x-x' \\ \\ y-y' \end{pmatrix} dx'dy'. \tag{97}$$

This formula can be used for practical numerical computations of two components of deflection of the vertical in the Cartesian coordinates.

The Molodenskii G_1 term is also expressed by a formula similar to Eq. (97), that is

$$G_1(x, y) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{h(x', y') - h(x, y)}{\{(x-x')^2+(y-y')^2\}^{3/2}} \delta g(x', y') dx'dy', \tag{98}$$

which corresponds to Eq. (57). The product of h and δg is defined as

$$\mu(x, y) = h(x, y)\delta g(x, y), \tag{99}$$

and then, using this new notation, Eq. (98) is divided into two integrals:

$$G_1(x, y) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{\mu(x', y') - \mu(x, y)}{\{(x-x')^2+(y-y')^2\}^{3/2}} dx'dy'$$

$$- \frac{h(x, y)}{2\pi} \iint_{-\infty}^{\infty} \frac{\delta g(x', y') - \delta g(x, y)}{\{(x-x')^2+(y-y')^2\}^{3/2}} dx'dy'. \tag{100}$$

It is readily noticeable that the first integral of the righthand side of Eq. (100) equals to G_{11} and the second one to G_{12}. The obtained Molodenskii terrain correction term is added to the deflection of the vertical calculated in the planar approximation. In rugged terrain areas the Molodenskii correction works efficiently for reducing observed values of deflection of the vertical to the surface of the ellipsoid.

It is found in this section that the planar approximation formulas in general have mathematically simple forms and are much easier than spherical—surface formulas to be dealt with. Very large sets of rectangular—gridded data can be used for the numerical solution of geodetic integral formulas within the limits of present data accuracy by means of a high—speed computation technique, such as FFT.

The disadvantage of the planar approximation method is, however, that, as long

—wavelength gravity anomalies of low—degree wavenumbers are excluded essentially from numerical solutions, calculated results (gravimetric deflection of the vertical) are not exactly equal to absolute values of astrogeodetic deflection of the vertical. Despite this disadvantage, short—wavelength undulations of local height anomaly are picked up well by the calculation. The calculated values of gravimetric deflection can preferably be used for the interpolation of absolute values astrogeodetically observed at the Laplace stations (triangulation points). The Molodenskii correction consideration may be neccessary for calculating deflections at triangulation points.

9. Conclusion

Recently several high—precision space techniques, such as VLBI (Very Long Baseline Interferometry), SLR (Satellite Laser Ranging) and GPS have been applied to geodetic measurements. In particular the portability of GPS receivers has made it possible to cover scales of 1—100km with dense geodetic networks. Three—dimensional vector separation can be obtained from GPS data, so that the GPS technique is now taking the place of both conventional triangulation and leveling techniques. Combination surveys of GPS and gravity have recently been conducted in mountainous areas in Japan because of the flexibility in selection of the stations.

The position of a point on the Earth's surface determined by the GPS technique directly refers to the geocentric coordinates of the Earth ellipsoid, whereas the leveling height is measured above the geoid and the latitude and longitude determined by triangulation surveys are based on a certain reference ellipsoid but not on the Earth ellipsoid. The GPS technique also has a large impact on gravity studies. The conventional geoid—based concept of gravity anomaly is now being replaced by this new satellite geodetic system, and another quantity, "gravity disturbance", will henceforth play an important role on physical geodesy instead of gravity anomaly.

This paper has treated the geodetic boundary—value problems according to the new geodetic system. In this case the Stokes and the Vening—Meinesz integral formulas are reformulated in the Neumann and the modified Vening—Meinesz integral formulas using gravity disturbance. We have discussed the inverse problem, in which gravity disturbance is inversely obtained from the height anomaly and evaluated the truncation error accompanied with numerical solutions. The Molodenskii terrain correction terms have also been discussed for high—precision calculations of gravimetric height anomaly and deflection of the vertical. Finally we have treated the planar approximation theory based on a flat—shaped Earth in the Cartesian coordinates.

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References

Denker, H. and H.—G. Wenzel (1987): Local geoid determination and comparison with GPS results, Bull. Beod., **61**, 349—366.

Engelis, T., Rapp, R. H. and Tscherning, C. C. (1984): The precise computation of geoid undulation differences with comparison to results obtained from the global positioning system, Geophys. Res. Lett., **1**, 821—824.

Forsberg, R. (1985): Gravity field terrain effect computations by FFT, Bull. Geod., **59**, 342—360.

Gleason, D. M. (1990): Obtaining earth surface and spatial deflections of the vertical from free—air gravity anomaly and elevation data without density assumptions, J. Geophys. Res., **95**, 6779—6786.

Hagiwara, Y. (1972): Truncation error formulas for the geoidal height and the deflection of the vertical, Bull. Geod., **46**, 453—466.

Hagiwara, Y. (1976): A new formula for evaluating the truncation error coefficient, Bull. Geod., **50**, 131—135.

Heiskanen, W. A. and H. Moritz (1967): Physical Geodesy, Freeman, San Francisco, p. 364.

Moritz, H. (1980): Advanced Physical Geodesy, Abacus, Tunbridge Wells, Kent, Great Britain, p. 500.

Molodenskii, M. S., Eremeev, V. F. and M. I. Yurkina (1962): Methods for Study of the External Gravitational Field and Figure of the Earth, Translated from the Russian by the Israel Prog. for Sc. Transl., Jerusalem. p. 248.

Saito, M. (1980): Molodenskii truncation error coefficients at high degrees——Stability of Hagiwara's formula and asymptotic formula, J. Geod. Soc. Japan, **26**, 180—186 (in Japanese).

Schwarz, K. P., Sideris, M. G. and R. Forsberg (1990): The use of FFT techniques in physical geodesy, Geophys. J. Int., **100**, 485—514.

Sideris, M. G. (1985): A fast Fourier transform method for computing terrain corrections, Manuscripta Geodetica, **10**, 66—73.

Sideris, M. G. and Schwarz, K. P. (1986): Solving Molodensky's series by fast Fourier transform techniques, Bull. Geod., **60**, 51—63.

Sideris, M. G. and Schwarz, K. P. (1988): Advances in the numerical solution of the linear Molodensky problem, Bull. Geod., **62**, 59—69.

Vassiliou, A. A. (1988): The computation of aliasing effects in local gravity field approximation, Bull. Geod., **62**, 41—58.

Zavattero, P. (1987): Computation of the singular components of the Vening—Meinesz integrals using the two—dimensional fast Fourier transform, Manuscripta Geodetica, **12**, 137—142.

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GPS を用いた重力計測のためのノイマン境界値問題の物理測地学

萩原幸男

近年 GPS により高精度に測地座標が決められるため，その技術は重力点の位置決めに応用されるようになった．GPS による座標は楕円体を基準にしたもので，従来の水準測量のようにジオイドを基準にしたものではない．そのためジオイドに立脚した従来の“重力異常”はその重要性を失い，かわって楕円体高に立脚した“重力乱れ”が登場することになった．いずれ重力異常をもとに構成された物理測地学も大幅な変更を与儀なくされることになろう．本論文はそれを見越した形で，ノイマン境界値問題に伴うノイマン積分と変形ベニングマイネス積分，その逆変換と切断誤差，またモロデンスキー補正項について議論した．さらに FFT への応用を考慮して，3 次元直角座標による近似についても言及した．

本論文の構成は 9 節よりなる．2 節において，重力異常との対比から，GPS 測量における

重力乱れの意義について述べ, 3 節においては, ノイマン境界値問題とその解について解説した. ここでは従来の“ジオイド高”ではなく, モロデンスキー測地学に従って“height anomaly”の用語を用いた. height anomaly の空間微分形としての垂直線偏差を, 4 節では, 変形ベニングマイネス積分を用いて記述した. 5 節では, ノイマン積分と変形ベニングマイネス積分の切断誤差の計算式を導出し, また 6 節では, 上記積分の逆問題を解いた. この 5 節と 6 節の内容は筆者の一連の研究の未発表の部分にあたる. これに対して, 7 節のモロデンスキー補正項は結果として従来の重力異常を重力乱れに置き換えたものとなるため, 内容的に新しいものとはいえない. また 8 節の直角座標表示も, 多少筆者独自の拡張はあるものの, レビュー的な内容が主である. それにも拘らず, ノイマン境界値問題を以上のように, 全体としてまとまった形式で記述しておくことは, GPS 時代に対処して, 今後予想される物理測地学の大幅な改訂に先駆けて意義あることと考える.